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Boundary value problems for differential forms on compact Riemannian manifolds, Part II

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Summary: This paper provides solutions to second order boundary value problems for differential forms by means of the method applied in [3] for first order problems. These r-forms $f \in C^{2,\lambda}(\bar{\Omega})$ have prescribed conditions $\triangle f$, where Ω is a subset of a compact Riemannian manifold and $\partial\Omega \neq \emptyset$. Further constraints are imposed by the boundary conditions and topological properties. Such boundary value problems play a prominent role in the proof of the Hodge–Kodaira–Morrey decomposition of L^2, as demonstrated in [9]. Our approach, based on the boundary integral method, will provide additional results, which will be essential parts of a direct proof of the corresponding Banach space decompositions of L^p.

1 Introduction

We present boundary value problems for r-forms with regard to the Laplace–Beltrami operator $\triangle$, where $0 < r < n$. These $\triangle$-Dirichlet problems are formally given by

$$\triangle f = g \text{ in } \Omega,$$

$$\nu \wedge f = \xi \text{ and } \nu \wedge \delta f = \vartheta \text{ on } \partial\Omega.$$

The forms g, ξ, and ϑ are assumed to satisfy particular regularity properties to provide solutions with components in $C^{2,\lambda}(\bar{\Omega})$. Here, Ω is a bounded set of a Riemannian manifold $\mathcal{M}$. If the form g belongs to the orthogonal complement of the L^2-completion of harmonic forms, the $\triangle$-Dirichlet problem can be solved. The dimension of the space of solutions is finite and is given by Betti numbers of the set Ω, which can be shown by Green's formula

$$\int_\Omega (\triangle f, g)\, dx = -\int_\Omega (\delta f, \delta g)\, dx - \int_\Omega (df, dg)\, dx + \int_{\partial\Omega} (\nu \wedge \delta f, g)\, d\omega + \int_{\partial\Omega} (\nu \wedge f, dg)\, d\omega,$$

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where f, g have to satisfy regularity properties. For $g = 0$, $\xi = 0$, and $\vartheta = 0$, Green's formula implies

$$df = 0, \quad \delta f = 0,$$
$$\nu \wedge f = 0.$$

Depending on the topology of the set Ω, the solution f might differ from 0. The dimension of this space of solutions is determined by the Betti number B_{n-r} of Ω.

The above mentioned $\triangle$ problem is widely discussed, for instance in [9]. For vector fields on $\Omega \subset \mathbb{R}^3$, this problem is treated e.g. in [7] and [11]. Duff deals with a related problem in [4] for $C^\infty(\mathcal{M}_\partial)$-forms, where $\mathcal{M}_\partial$ is assumed to be a Riemannian manifold with boundary. In [9], Morrey used a weak formulation of this problem in order to find an analogy to the Hodge–Kodaira decomposition for $\mathcal{M}_\partial$. This approach provides a decomposition of the Hilbert space $L^2(\mathcal{M}_\partial)$.

Our concept, however, presents a direct access to decompositions for the Banach spaces $C^{0,\lambda}$ and L^p. This can be achieved by potential theoretical results, the boundary integral method, and results presented by Weyl in [13], where an analogous boundary value problem concerning the wave equation operator $\triangle + k^2$ and skew-symmetrical tensor fields in Euclidean space is treated. For the Euclidean space, the proofs are performed in [2].

We define forms $f = f_\phi$ depending on forms $\phi(.)$ and double forms $G_\mathcal{M}(\cdot, \cdot)$. This double form is the kernel of an operator $G_\mathcal{M}$ de Rahm [10] has introduced. By means of an additional projection $H_\mathcal{M}$, he shows that

$$I - H_\mathcal{M} = \triangle G_\mathcal{M}.$$

For the form ϕ defined on $\partial\Omega$, we derive the inhomogeneous integral equation

$$\left(\pm \frac{1}{2} I + W_r \right) \phi = \begin{pmatrix} \nu \lrcorner (\nu \wedge f_\mp) \\ \nu \lrcorner (\nu \wedge \delta f_\mp) \end{pmatrix},$$

where f_- and f_+ are given by approximations in Ω and $\hat{\Omega}$, respectively. By using the boundary integral method, we prove that suitable operators G_Ω^+ and G_Ω^- exist to solve the $\triangle$-Dirichlet problem and the dual $\triangle$-Neumann problem for homogeneous boundary data. Analogously to the approach in [2] for the more special case of Euclidean spaces, it is thereupon an easy task to proof the decomposition

$$I - H_\Omega = d\delta G_\Omega^+ + \delta d G_\Omega^-.$$

The operator H_Ω is the restriction of $H_\mathcal{M}$ to $\bar{\Omega}$.

For further investigations, it might be one of the next steps to generalize the results of Kress in [7] to r-forms and $\mathcal{M}_\partial$. His method is based on a representation for vector-fields on $\Omega \subset \mathbb{R}^3$ reflecting both the boundary values of a $\triangle$-Dirichlet problem and a $\triangle$-Neumann problem. For domains in $\mathbb{R}^3$, he derives the integral equations

$$(I - B) \begin{pmatrix} \omega^* \\ \epsilon^* \end{pmatrix} = \gamma,$$

where B are linear operators, ω^* are vector fields, and ϵ^* are scalars.

As the Hodge–Kodaira–Morrey decomposition for vector fields in $\Omega \subset \mathbb{R}^3$ yields a refined Helmholtz decomposition (cf. [6, 12]), there will be a bunch of further generalizations with regard to unbounded manifolds.

2 Definitions and preliminaries

Definition 2.1 *For $n \in \mathbb{N} : n \geq 2$, let $\mathcal{M} = \mathcal{M}^n$ be a compact, oriented n dimensional C^∞-Riemannian manifold. Let $\Omega_i = \Omega_i^n \subset\subset \mathcal{M}$ be oriented n dimensional, arcwise connected C^∞-Riemannian manifolds with $\Omega_i \neq 0$ which satisfy the conditions*

$$\bar{\Omega}_i \cap \bar{\Omega}_j = \emptyset \quad \text{if } i \neq j \text{ and if } \partial\Omega_i \text{ belongs to } C^\infty.$$

The set Ω is the union of a finite number of Ω_i.

Furthermore, $\hat{\Omega}$ designates the set $\mathcal{M}\backslash\bar{\Omega}$. The exterior normal 1-form is denoted by ν.

According to the common notation, d designates the exterior derivative, and δ the coderivative, which can be expressed by d and the Hodge mapping $*$. For C^1-forms $f = f_r$, we set

$$\delta f := (-1)^{(r+1)n} * (d(*f)) \quad \text{if } 0 < r \leq n$$

$$\text{and } \delta f = 0 \quad \text{if } r = 0.$$

If f, h are r-forms, where $0 < r \leq n$, then the inner product (f, h) is defined by

$$(f, h) = (h, f) := \frac{1}{r!} \sum_{j_1 \dots j_r, l_1 \dots l_r} g^{j_1 l_1} \dots g^{j_r l_r} f_{J^r} h_{L^r}.$$

For 0-forms f and h, we define $(f, h) = (h, f) := f \cdot h$.

Now, let γ be a 1-form, f be an r-form, and $\lrcorner$ be the contraction. For $0 < r \leq n$, the $(r-1)$-form $\gamma \lrcorner f$ is given in local coordinates by

$$\gamma \lrcorner f = f \lrcorner \gamma := \sum_{l_2 < \dots < l_r} (\gamma \lrcorner f)_{l_2 \dots l_r} dx^{l_2} \wedge \dots \wedge dx^{l_r}, \text{ where}$$

$$(\gamma \lrcorner f)_{l_2 \dots l_r} := \sum_{k,l=1}^{n} \gamma_k g^{kl} f_{l l_2 \dots l_r}.$$

If f is a 0-form, then we set $\gamma \lrcorner f = f \lrcorner \gamma := 0$.

For a linear space L, the symbol L^r denotes the space of r-forms with components in L.

We assume that $k \in \mathbb{N}_0$ and $0 < \lambda < 1$ for spaces $C^{k,\lambda}$. The expression $C^{k,\lambda}(\partial\Omega)^r$ is used as an abbreviation for $C^{k,\lambda}(\bar{\Omega})^r{}_{|\partial\Omega}$. Analogously, spaces C^k are given.

For a differential form f defined on $\partial\Omega$, we call $\nu \lrcorner f$ the normal component of f and $\nu \wedge f$ the tangential component of f. Subscripts ν or τ mean that the space in question consists of elements with vanishing normal or tangential components respectively.

The spaces $L^q(\partial\Omega)^r$, $1 \leq q < \infty$, denote the L^q-completion of $C^\infty(\partial\Omega)^r$. Analogously, the spaces $L^q_\nu(\partial\Omega)^r$ and $L^q_\tau(\partial\Omega)^r$ are defined.

By $R_{r,r}(x, y)$ double forms are denoted, where

$$R_{r,r}(x, y) = \sum_{i_1<\cdots<i_r} \sum_{j_1<\cdots<j_r} R_{i_1\ldots i_r, j_1\ldots j_r} \left(dx^{i_1} \wedge \cdots \wedge dx^{i_r}\right) \otimes \left(dy^{i_1} \wedge \cdots \wedge dy^{i_r}\right).$$

In the following sections, double forms are kernels of integral operators $O_{k,\lambda,r}$ for instance, which map r-forms with components in $C^{k,\lambda}$ to r'-forms. Furthermore, $O_{2,r}$ are L^2-extensions of operators in C^∞.

If the domains of theses operators are already clarified by the context, the notation O_r for such operators can also be used.

3 Boundary value problems for differential forms

In this section, we solve integral equations for differential forms in order to find solutions to our boundary value problems and to deduce a priori inequalities. The following results serve to prove these estimates.

By means of [10, 1], we can assert that there exist operators

$$G \in \mathcal{L}\left(C^{k,\lambda}(\mathcal{M})^r, C^{k+2,\lambda}(\mathcal{M})^r\right) \text{ and } H \in \mathcal{L}\left(C^{k,\lambda}(\mathcal{M})^r, C^{k,\lambda}(\mathcal{M})^r\right),$$

constituted by symmetric kernels $G_{r,r}(x, y)$ and $H_{r,r}(x, y)$, which satisfy

$$\triangle G\phi = \phi - H\phi \quad \text{for all } \phi \in C^\infty(\mathcal{M})^r. \tag{3.1}$$

G and H are their own metric transpose with regard to the L^2 extensions and the inner product

$$\langle\cdot,\cdot\rangle : L^2(\partial\Omega)^r \times L^2(\partial\Omega)^r \longrightarrow \mathbb{R},$$

given by

$$\langle f, g\rangle := \int_{\partial\Omega} (f(x), g(x))_x \, d\omega_x.$$

Additionally, G and H commute with $*$. Further properties are stated in the above mentioned publications.

In order to solve our boundary value problem, we search for solutions to related Fredholm integral equations for differential forms over $\partial\Omega$. The following lemma offers the first step to define these integral operators.

Lemma 3.1 *We assume that $f = f_r \in C^{k,\lambda}_\nu(\partial\Omega)^r$ and $x, y \in \partial\Omega$, and define operators $U_{k,\lambda,r}$, $V_{k,\lambda,r}$, $U^*_{k,\lambda,r}$, and $V^*_{k,\lambda,r}$ as*

$$U_{k,\lambda,r} f_r(x) := \int_{\partial\Omega} \nu(x) \lrcorner (\nu(x) \wedge (d_y G_{r,r}(x, y), \nu(y) \wedge f_r(y))_y) \, d\omega_y,$$

$$V_{k,\lambda,r} f_r(x) := \int_{\partial\Omega} \nu(x) \lrcorner (\nu(x) \wedge (G_{r+1,r+1}(x, y), \nu(y) \wedge f_r(y))_y) \, d\omega_y,$$

$$U^*_{k,\lambda,r} f_r(x) := \int_{\partial\Omega} \nu(x) \lrcorner (d_x G_{r,r}(x,y),\ f_r(y))_y \, d\omega_y,$$
$$V^*_{k,\lambda,r} f_r(x) := \int_{\partial\Omega} \nu(x) \lrcorner (G_{r,r}(x,y),\ f_r(y))_y \, d\omega_y.$$

These linear operators are compact in $C^{k,\lambda}_\nu(\partial\Omega)^r$ *and*

$$U_{k,\lambda,r} \in \mathcal{L}\left(C^{k,\lambda}_\nu(\partial\Omega)^r, C^{k+1,\lambda}_\nu(\partial\Omega)^r\right), \quad V_{k,\lambda,r} \in \mathcal{L}\left(C^{k,\lambda}_\nu(\partial\Omega)^r, C^{k+1,\lambda}_\nu(\partial\Omega)^{r+1}\right),$$

$$U^*_{k,\lambda,r} \in \mathcal{L}\left(C^{k,\lambda}_\nu(\partial\Omega)^r, C^{k+1,\lambda}_\nu(\partial\Omega)^r\right), \quad V^*_{k,\lambda,r} \in \mathcal{L}\left(C^{k,\lambda}_\nu(\partial\Omega)^r, C^{k+1,\lambda}_\nu(\partial\Omega)^{r-1}\right).$$

Proof: The proof for $V_{k,\lambda,r}$ and $V^*_{k,\lambda,r}$ is evident. For the assertion with regard to the operators $U_{k,\lambda,r}$ and $U^*_{k,\lambda,r}$, it is convenient to refer to the operators

$$K_r = K_{k,\lambda,r} \in \mathcal{L}\left(C^{k,\lambda}_\nu(\partial\Omega)^r, C^{k+1,\lambda}_\nu(\partial\Omega)^r\right)$$

and

$$L_r = L_{k,\lambda,r} \in \mathcal{L}\left(C^{k,\lambda}_\tau(\partial\Omega)^r, C^{k+1,\lambda}_\tau(\partial\Omega)^r\right),$$

given by

$$(K_r f)(x) := -2 \int_{\partial\Omega} \nu(x) \lrcorner (d_x G_{r,r}(x,y),\ f(y))_y \, d\omega_y, \text{ where } 0 \le r < n,$$

and

$$(L_r f)(x) := -2 \int_{\partial\Omega} \nu(x) \wedge (\delta_x G_{r,r}(x,y),\ f(y))_y \, d\omega_y, \text{ where } 0 < r \le n,$$

cf. [3]. As the operators U^*_n and U_n vanish, $0 \le r < n$ can be assumed. We take into account that U^*_r is related to K_r by

$$U^*_r f = -\frac{1}{2} K_r f, \tag{3.2}$$

and U_r to L_{r+1} by

$$U_r f = \frac{1}{2} \nu \lrcorner L_{r+1}(\nu \wedge f). \tag{3.3}$$

Alternatively, equation (3.17) can be used. □

Referring to the operators of *Lemma 3.1*, we can define the operators $W = W_r$ and $W = W^*_r$, which appear in the jump conditions related to our boundary problems.

Definition 3.2 *Let* $0 < r \le n$. *We set*

$$W_{k_1,k_2,\lambda,r} := \begin{pmatrix} U_{k_1,\lambda,r} & -V_{k_2,\lambda,r-1} \\ 0 & U_{k_2,\lambda,r-1} \end{pmatrix} \quad \textit{and}$$
$$W^*_{k_1,k_2,\lambda,r} := \begin{pmatrix} U^*_{k_1,\lambda,r} & 0 \\ -V^*_{k_1,\lambda,r} & U^*_{k_2,\lambda,r-1} \end{pmatrix}.$$

Lemma 3.3 *The operators $U_{2,r}$ and $U^*_{2,r}$ as well as the operators $V_{2,r}$ and $V^*_{2,r+1}$ are adjoint operators, i.e. metric transposes, with respect to the inner product*

$$\langle \cdot, \cdot \rangle : L^2(\partial\Omega)^r \times L^2(\partial\Omega)^r \longrightarrow \mathbb{R},$$

given above.

Note: As $U_{2,r}$ and $V_{2,r}$ are defined only for forms in $L^2_\nu(\partial\Omega)^r$, a restricted inner product with regard to this differential forms would suffice.

Proof: Since U_n, U^*_n, V_n, and V^*_{n+1} vanish, we confine ourselves to $0 \le r < n$.
(i) For the further steps, it is useful to mention the formulas (3.4) and (3.5). Inner products and contractions are performed pointwise there.

If $e = e_r$, $h = h_{r+1}$ and $u = u_1$, i.e. e is assumed to be an r-form, h an $(r+1)$-form, and u a 1-form, contraction and exterior product are connected by:

$$(e, u \lrcorner h) = (u \wedge e, h). \tag{3.4}$$

Let $p = p_l$, $u = u_1$, and $v = v_1$. The term $p(u, v)$ can be decomposed into:

$$p\,(u, v) = u \wedge (v \lrcorner p) + (v \lrcorner u \wedge p). \tag{3.5}$$

(ii) In this section, we will show that $V^*_{2,r+1}$ is the metric transpose of $V_{2,r}$. Let $f \in L^2_\nu(\partial\Omega)^r$ and $g \in L^2_\nu(\partial\Omega)^{r+1}$. It is not necessary to prove the assertion for the case $r+1 = n$, as the requirement $\nu \lrcorner g = 0$ for the n-form g implies that g vanishes.

Then, the relevant inner product can be written as

$$\begin{aligned} &\int_{\partial\Omega} ((V_r f)(x), g(x))_x \, d\omega_x \qquad (3.6)\\ &= \int_{\partial\Omega}\int_{\partial\Omega} \nu(x) \lrcorner (\nu(x) \wedge (G_{r+1,r+1}(x, y), \nu(y) \wedge f_r(y))_y, g_{r+1}(x))_x \, d\omega_y \, d\omega_x. \end{aligned}$$

For the form $\mu = \mu(x, y) := (G_{r+1,r+1}(x, y), \nu(y) \wedge f_r(y))_y$, we obtain by using (3.4):

$$(\nu \lrcorner (\nu \wedge \mu), g) = (\mu, \nu \lrcorner (\nu \wedge g)). \tag{3.7}$$

As $\nu \lrcorner g = 0$ and $\|\nu\| = 1$, equations (3.5) and (3.7) imply

$$(\nu \lrcorner (\nu \wedge \mu), g) = (\mu, g). \tag{3.8}$$

Hence, we conclude that

$$\begin{aligned} &\int_{\partial\Omega} ((V_r f)(x), g(x))_x \, d\omega_x \\ &= \int_{\partial\Omega}\int_{\partial\Omega} ((G_{r+1,r+1}(x, y), \nu(y) \wedge f_r(y))_y, g_{r+1}(x))_x \, d\omega_y \, d\omega_x \\ &= \int_{\partial\Omega} \left(\left(\int_{\partial\Omega} \nu(y) \lrcorner (G_{r+1,r+1}(x, y), g_{r+1}(x))_x \, d\omega_x\right), f_r(y)\right)_y d\omega_y \\ &= \int_{\partial\Omega} (f_r(y), (V^*_{r+1} g_{r+1})(y))_y \, d\omega_y. \end{aligned} \tag{3.9}$$

This means that

$$\langle (V_r f), g\rangle = \langle f, (V^*_{r+1} g)\rangle, \tag{3.10}$$

and $V^*_{2,r+1}$ is the adjoint operator of $V_{2,r}$.

(iii) In the same manner, we will show that $U^*_{2,r}$ is the metric transpose of $U_{2,r}$.

Let $f \in L^2_\nu(\partial\Omega)^r$ and $g \in L^2_\nu(\partial\Omega)^r$. Then,

$$\int_{\partial\Omega} ((U_r f)(x), g(x))\, d\omega_x$$
$$= \int_{\partial\Omega}\int_{\partial\Omega} \nu(x) \lrcorner (\nu(x) \wedge (d_y G_{r,r}(x,y), \nu(y) \wedge f_r(y))_y, g_r(x))_x\, d\omega_y\, d\omega_x. \tag{3.11}$$

For $\gamma = \gamma(x,y) := (d_y G_{r,r}(x,y), \nu(y) \wedge f_r(y))_y$, we obtain analogously to (3.8):

$$(\nu \lrcorner (\nu \wedge \gamma), g) = (\gamma, g). \tag{3.12}$$

The remaining part of the assertion can be proved by

$$\begin{aligned}
&\int_{\partial\Omega} ((U_r f)(x), g(x))\, d\omega_x \\
&= \int_{\partial\Omega}\int_{\partial\Omega} ((d_y G_{r,r}(x,y), \nu(y) \wedge f_r(y))_y, g_r(x))_x\, d\omega_y\, d\omega_x \\
&= \int_{\partial\Omega}\int_{\partial\Omega} ((d_y G_{r,r}(x,y), g_r(x))_x, \nu(y) \wedge f_r(y))_y\, d\omega_y\, d\omega_x \\
&= \int_{\partial\Omega} \left(\int_{\partial\Omega} (\nu(y) \lrcorner (d_y G_{r,r}(x,y), g_r(x))_x)\, d\omega_x,\ f_r(y) \right)_y d\omega_y \\
&= \int_{\partial\Omega} (f(y), (U^*_r g)(y))_y\, d\omega_y.
\end{aligned} \tag{3.13}$$

This yields

$$\langle (U_r f), g\rangle = \langle f, (U^*_r g)\rangle, \tag{3.14}$$

and $U^*_{2,r}$ is the adjoint operator of $U_{2,r}$. □

The operator K_r, where $0 < r \le n$, can be considered to be a composition of L_{n-r} and the Hodge mapping:

$$K_r *_y = *_x L_{n-r}, \tag{3.15}$$

cf. [3, 5], and [8]. By calling $*^{-1} O *$ the topological transpose of an operator O, we can express this property in an even more concise manner.

For the metric transpose, the following assertion is stated there: With regard to the bilinear form

$$\langle \cdot, \cdot \rangle : L^2_\nu(\partial\Omega)^r \times L^2_\tau(\partial\Omega)^r \longrightarrow \mathbb{R},$$

given by

$$\langle f, g\rangle := \int_{\partial\Omega} (f(x), (\nu(x) \lrcorner g(x))\, d\omega_x,$$

where $0 \le r < n$, $-L_{r+1}$ is the adjoint operator of K_r.

In the following lemma, the topological transposes of $V^*_{n-(r+1)}$ and $U^*_{n-(r+1)}$ are investigated.

Lemma 3.4 *Let $f_r \in L^2_\nu(\partial\Omega)^r$ and $0 \le r < n$. Then,*

$$a)\quad V_r f_r = (-1)^{r+1} \nu \lrcorner \left(*_x^{-1} V^*_{n-(r+1)} *_y (\nu \wedge f_r) \right) \tag{3.16}$$

$$b)\quad U_r f_r = -\nu \lrcorner \left(*_x^{-1} U^*_{n-(r+1)} *_y (\nu \wedge f_r) \right). \tag{3.17}$$

Proof: a) The operators G and $*$ commute, as shown in [10]. This means that

$$*_y^{-1} G_{n-(r+1),n-(r+1)} = *_x G_{r+1,r+1}, \tag{3.18}$$

which, with regard to the operator $V^*_{n-(r+1)}$, implies the relationship

$$\begin{aligned} &\left(V^*_{n-(r+1)} *_y (\nu \wedge f_r) \right)(x) \\ &= \int_{\partial\Omega} \nu(x) \lrcorner \left(*_y^{-1} G_{n-(r+1),n-(r+1)}(x, y), (\nu \wedge f_r)(y) \right)_y d\omega_y \\ &= \int_{\partial\Omega} \nu(x) \lrcorner *_x (G_{r+1,r+1}(x, y), (\nu \wedge f_r)(y))_y \, d\omega_y. \end{aligned} \tag{3.19}$$

From equation

$$\nu \wedge *f_r = (-1)^{r-1} * (\nu \lrcorner f_r), \tag{3.20}$$

we thus derive

$$V_r f_r = (-1)^{r+1} \nu \lrcorner \left(*_x^{-1} V^*_{n-(r+1)} *_y (\nu \wedge f_r) \right). \tag{3.21}$$

b) The operators L_{r+1} and $K_{n-(r+1)}$ can be deployed in order to prove an equation connecting U_r and $U^*_{n-(r+1)}$. From (3.15) and (3.2), we obtain

$$*_x L_{r+1}(\nu \wedge f_r) = K_{n-(r+1)}(*_y(\nu \wedge f_r)) = -2U^*(*_y(\nu \wedge f_r)). \tag{3.22}$$

Additionally, (3.20) yields

$$*_x(\nu, L_{r+1}(\nu \wedge f_r)) = (-1)^r \nu \wedge (*_y L_{r+1}(\nu \wedge f_r)). \tag{3.23}$$

Then (3.3), (3.23), and (3.22) imply

$$\begin{aligned} *_x U_r f_r &= \frac{1}{2}(-1)^r \nu \wedge (*_y L_{r+1}(\nu \wedge f_r)) \\ &= -(-1)^r \nu \wedge \left(U^*_{n-(r+1)} *_y (\nu \wedge f_r) \right). \end{aligned} \tag{3.24}$$

This finally provides

$$U_r f_r = -\nu \lrcorner \left(*_x^{-1} U^*_{n-(r+1)} *_y (\nu \wedge f_r) \right). \tag{3.25}$$

□

The following theorem presents particular solutions to $\mathcal{R}((\pm\frac{1}{2}I + W_r))$ and $\mathcal{R}((\mp\frac{1}{2}I + W_r^*))$. This result is based on ideas of H. Weyl for subsets of the manifold $\mathbb{R}^n$, which can be found in [13].

Theorem 3.5 *Let* $0 < r < n$ *and* $x \in \mathcal{M}\backslash\partial\Omega$.

We presuppose that $\psi \in C_\nu^{0,\lambda}(\partial\Omega)^r$ *and* $\varphi \in C_\nu^{0,\lambda}(\partial\Omega)^{r-1}$, $\phi := (\psi, \varphi)$, *and*

$$f_{r,\phi}(x) := f_{r,(0,\varphi)}(x) + f_{r,(\psi,0)}(x),$$

where

$$f_{r,(0,\varphi)}(x) := -\int_{\partial\Omega} G_{r,r}(x, y) \wedge *\varphi_{r-1}(y)$$

and

$$f_{r,(\psi,0)}(x) := \int_{\partial\Omega} d_y G_{r,r}(x, y) \wedge *\psi_r(y).$$

Furthermore, let $\varphi^* \in C_\nu^{0,\lambda}(\partial\Omega)^r$, $\psi^* \in C_\nu^{0,\lambda}(\partial\Omega)^{r-1}$, $\phi^* := (\varphi^*, \psi^*)$, *and*

$$f^*_{r,\phi^*}(x) := f^*_{r,(\varphi^*,0)}(x) + f^*_{r,(0,\psi^*)}(x),$$

where

$$f^*_{r,(\varphi^*,0)}(x) := \int_{\partial\Omega} \big(G_{r,r}(x, y), \varphi_r^*(y)\big)_y \, d\omega_y$$

and

$$f^*_{r,(0,\psi^*)}(x) := -\int_{\partial\Omega} \big(d_x G_{r-1,r-1}(x, y), \psi_{r-1}^*(y)\big)_y \, d\omega_y.$$

The r*-forms* $f_{r,\phi}$ *and* f^*_{r,ϕ^*} *are harmonic differential forms, i.e.*

$$H f_{r,\phi} = 0 \quad \text{and} \quad H f^*_{r,\phi^*} = 0.$$

Additionally, ϕ *is a solution to the inhomogeneous integral equation*

$$\Big(\pm\frac{1}{2}I + W_r\Big)\phi = \begin{pmatrix} \nu \lrcorner (\nu \wedge f) \\ \nu \lrcorner (\nu \wedge \delta f) \end{pmatrix}$$

and ϕ^* *is a solution to the inhomogeneous integral equation*

$$\Big(\mp\frac{1}{2}I + W_r^*\Big)\phi^* = \begin{pmatrix} \nu \lrcorner\, d f^* \\ -\nu \lrcorner\, f^* \end{pmatrix}.$$

The upper (lower) signs are used if the approximation is performed in Ω ($\hat{\Omega}$) *along* $+\nu$ ($-\nu$).

Proof: For the moment, we consider the set $\mathcal{M}\backslash\partial\Omega$ and presuppose that points x are contained in it. The differential forms $f_{r,\phi}$ and f^*_{r,ϕ^*} are C^∞-forms here.

By means of

$$\delta_x G_{r+1,r+1} = -d_y G_{r,r}, \tag{3.26}$$

from [10], the equation

$$\begin{aligned} f_{r,(\psi,0)}(x) &= -\int_{\partial\Omega} (\delta_x G_{r+1,r+1})(x,y) \wedge *\psi_r(y) \\ &= -\delta_x \int_{\partial\Omega} G_{r+1,r+1}(x,y) \wedge *\psi_r(y) \end{aligned} \tag{3.27}$$

follows. Hence, $f_{r,(\psi,0)}$ is co-exact, i.e. it can be expressed by the codifferential of an $(r+1)$-form. This and (3.26) imply that the codifferential of $f_{r,\phi}$,

$$\delta f_{r,\phi}(x) = \delta f_{r,(0,\varphi)}(x) = \int_{\partial\Omega} d_y G_{r-1,r-1}(x,y) \wedge *\varphi_{r-1}(y). \tag{3.28}$$

Analogously, the form $f^*_{r,(0,\psi^*)}$ is exact, which can be shown by

$$\begin{aligned} f^*_{r,(0,\psi^*)}(x) &= -\int_{\partial\Omega} \left(d_x G_{r-1,r-1}(x,y), \psi^*_{r-1}(y)\right)_y \, d\omega_y \\ &= -d_x \int_{\partial\Omega} \left(G_{r-1,r-1}(x,y), \psi^*_{r-1}(y)\right)_y \, d\omega_y. \end{aligned} \tag{3.29}$$

Therefore, the exterior differential of f^*_{r,ϕ^*} is given by

$$d f^*_{r,\phi^*}(x) = \int_{\partial\Omega} \left(d_x G_{r,r}(x,y), \varphi^*_r(y)\right)_y \, d\omega_y. \tag{3.30}$$

From [10], we know that $HG = 0$ or equivalently

$$\int_{y\in\mathcal{M}} H(x,y) \wedge *_y\, G(y,z) = 0. \tag{3.31}$$

Hence, the differential forms $f_{r,\phi}$ and f^*_{r,ϕ^*} are harmonic, i.e.

$$H f_{r,\phi} = 0 \text{ and } H f^*_{r,\phi^*} = 0. \tag{3.32}$$

Let υ be a suitable extension of ν, and let x be a point outside of the set $\partial\Omega$. The operators U_r, V_r, U^*_r and V^*_r are analogously defined with regard to $\upsilon(x)$ instead of $\nu(x)$.

Our jump conditions can be derived by expressing $f_{r,\phi}(x)$ and $f^*_{r,\phi^*}(x)$ in terms of the operators from *Lemma 3.1*, where $x \in \Omega$. For the form $f_{r,\phi}$, the following equations are valid:

$$\upsilon(x) \lrcorner (\upsilon(x) \wedge f_{r,\phi}(x)) = -(V_{r-1}\varphi)(x) + (U_r\psi)(x) \tag{3.33}$$

and

$$\upsilon(x)\lrcorner\,(\upsilon(x)\wedge\delta f_{r,\phi}(x)) = (U_{r-1}\varphi)(x). \tag{3.34}$$

Correspondingly, we obtain for the form f^*_{r,ϕ^*}:

$$\upsilon(x)\lrcorner\, f^*_{r,\phi^*}(x) = \left(V^*_r\varphi^*\right)(x) - \left(U^*_{r-1}\psi^*\right)(x) \tag{3.35}$$

and

$$\upsilon(x)\lrcorner\, d f^*_{r,\phi^*}(x) = \left(U^*_r\varphi^*\right)(x). \tag{3.36}$$

The relevant jump conditions can be taken from [3] or [5]. Differential forms u_r given by

$$u_r(x) := \int_{\partial\Omega} G_{r,r}(x,y)\wedge *_y\eta_{r+1}(y), \tag{3.37}$$

have the properties

$$du_{\mp}(x) = \int_{\partial\Omega} d_x G_{r,r}(x,y)\wedge *_y\eta(y) \mp \frac{1}{2}\,(\nu\wedge(\nu\lrcorner\,\eta))(x),\ \ x\in\partial\Omega, \tag{3.38}$$

where $\eta\in C^{0,\lambda}(\bar{\Omega})^{r+1}$ is required for u_r on Ω and $\eta\in C^{0,\lambda}(\bar{\hat{\Omega}})^{r+1}$ for u_r on $\hat{\Omega}$. The indices $-$ and $+$ mean that the approximation is performed in Ω and $\hat{\Omega}$, respectively. For a detailed proof, the reader is referred to [3].

From (3.38), we easily obtain the jump property for $\nu\lrcorner\, du$,

$$\begin{aligned}(\nu\lrcorner\, du_{\mp})(x) \;=\; &\int_{\partial\Omega} ((\nu(x)\lrcorner\, d_x G_{r,r}(x,y)), (\nu\lrcorner\,\eta)(y))_y\, d\omega_y \\ &\mp \frac{1}{2}\,(\nu\lrcorner\,\eta)(x).\end{aligned} \tag{3.39}$$

Using the operator U^*, this yields

$$(U^*h)_{\mp}(x) = (U^*h)(x) \mp \frac{1}{2}\,h(x). \tag{3.40}$$

Now, we are able to formulate our integral equations with regard to $\nu\lrcorner\, f^*_{r,\phi^*}$,

$$\left(\nu\lrcorner\, f^*_{r,\phi^*}\right)_{\mp}(x) = \pm\frac{1}{2}\,\psi^*_{r-1}(x) - \left(U^*_{r-1}\psi^*_{r-1}\right)(x) + \left(V^*_r\varphi^*_r\right)(x), \tag{3.41}$$

and to $\nu\lrcorner\, d f^*_{r,\phi^*}$,

$$\left(\nu\lrcorner\, d f^*_{r,\phi^*}\right)_{\mp}(x) = \mp\frac{1}{2}\,\varphi^*_r(x) + \left(U^*_r\varphi^*_r\right)(x). \tag{3.42}$$

Beside these requirements which ϕ^* has to fulfill, we are looking for constraints on ϕ with regard to $\nu\lrcorner\,(\nu\wedge f_{r,\phi})$ and $\nu\lrcorner\,(\nu\wedge\delta f_{r,\phi})$.

Let $\gamma \in C^{0,\lambda}(\bar{\Omega})^r$ or $\gamma \in C^{0,\lambda}(\bar{\hat{\Omega}})^r$, and

$$w_{r+1}(x) := \int_{\partial\Omega} G_{r+1,r+1}(x,y) \wedge *_y \gamma_r(y). \tag{3.43}$$

The form $\delta w_{\mp}$ is represented by

$$\delta w_{\mp}(x) = \int_{\partial\Omega} \delta_x G_{r+1,r+1}(x,y) \wedge *_y \gamma(y) \mp \frac{1}{2}\,(\nu \lrcorner (\nu \wedge \gamma))(x), \quad x \in \partial\Omega. \tag{3.44}$$

This implies the jump property for $\nu \lrcorner (\nu \wedge \delta w)$,

$$(\nu \lrcorner (\nu \wedge \delta w_{\mp}))(x) = -\int_{\partial\Omega} ((\nu(x) \lrcorner (\nu(x) \wedge d_y G_{r,r}(x,y))), (\nu \wedge \gamma)(y))_y \, d\omega_y \mp \frac{1}{2}\,\gamma(x). \tag{3.45}$$

Using the operator *U*, we obtain

$$(Uh)_{\mp}(x) = (Uh)(x) \pm \frac{1}{2}\,h(x). \tag{3.46}$$

Hence, the integral equation with regard to $\nu \lrcorner (\nu \wedge f_{r,\phi})$,

$$(\nu \lrcorner (\nu \wedge f_{r,\phi}))_{\mp}(x) = \pm\frac{1}{2}\,\psi_r(x) + (U_r\psi_r)(x) - (V_{r-1}\varphi_{r-1})(x), \tag{3.47}$$

and to $\nu \lrcorner (\nu \wedge d f^*_{r,\phi})$,

$$\left(\nu \lrcorner \left(\nu \wedge \delta f^*_{r,\phi}\right)\right)_{\mp}(x) = \pm\frac{1}{2}\,\varphi_{r-1}(x) + (U_{r-1}\varphi_{r-1})(x) \tag{3.48}$$

are derived. □

The results of [2] for subsets of the Euclidean space $\mathbb{R}^n$ are included in the approach presented here. This becomes obvious when

$$V_{r-1},\, V_r^*,\, \varphi_{r-1},\, \psi_{r-1}^*$$

are multiplied by $(-1)^{r-1}$ in order to obtain the corresponding definitions there. The operators

$$U_r,\, U_{r-1}^*,\, \psi_r,\, \varphi_r^*$$

are equally defined in both papers.

Harmonic fields and fields with additional vanishing normal or tangential component play an important role in the solution of our boundary value problems. Therefore, the following definition presents some basic quantities.

Definition 3.6 (Dirichlet and Neumann fields) *Let $0 < r < n$. Harmonic fields $f \in C^{1,\lambda}(\bar{\Omega})^r$ which satisfy $\nu \lrcorner f = 0$ are called Neumann fields $\mathcal{Z}(\Omega)^r$ on Ω. Those harmonic fields $f \in C^{1,\lambda}(\bar{\Omega})^r$ which satisfy $\nu \wedge f = 0$ are denoted as Dirichlet fields $\mathcal{Y}(\Omega)^r$ on Ω. The same applies with regard to $\hat{\Omega}$.*

$B_r = B_r(\Omega)$ refers to the Betti number of order r with regard to the set Ω.

Let $f \in L^1(\partial\Omega)^r$, $\{\hat{z}^i\}_{i=1,\dots,B_{n-r}}$ be a basis of $\mathcal{Z}(\hat{\Omega})^{r-1}$, and $\{\hat{y}^i\}_{i=1,\dots,B_r}$ be a basis of $\mathcal{Y}(\hat{\Omega})^{r+1}$. The topological quantities $Y_i[f]$ and $Z_i[f]$ are given by

$$Y_i[f] := -\int_{\partial\Omega} \left(f, \nu \wedge \hat{z}^i \right) d\omega, \ i = 1, \dots, B_{n-r},$$

and

$$Z_i[f] := -\int_{\partial\Omega} \left(f, \nu \lrcorner \hat{y}^i \right) d\omega, \ i = 1, \dots, B_r.$$

Both values $Y_i[f]$ and $Z_i[*f]$ differ merely by a factor $c := (-1)^{n(r+1)}$. This can be shown by

$$Y_i[f] := -\int_{\partial\Omega} \left(*f, *\left(\nu \wedge \hat{z}^i \right) \right) d\omega = -\int_{\partial\Omega} \left(*f, *\left(\nu \wedge *\hat{y}^i \right) \right) d\omega \qquad (3.49)$$

$$= -(-1)^{n-r} \int_{\partial\Omega} \left(*f, **\left(\nu \lrcorner \hat{y}^i \right) \right) d\omega = cZ_i[*f]. \qquad (3.50)$$

The boundary value problem for the Laplace–Beltrami operator can be uniquely solved if a topological constraint is taken into account. In [2], such boundary value problems for $\Omega \subset \mathbb{R}^n$ are formulated. We will repeat this approach here in the more general context of Riemannian manifolds.

Definition 3.7 (Dirichlet and Neumann problems for Δ) *Let $0 < r < n$.*
*a) The **Δ_r-Dirichlet problem**: We presuppose that $\xi \in C^{2,\lambda}_\tau(\partial\Omega)^{r+1}$, $\vartheta \in C^{1,\lambda}_\tau(\partial\Omega)^r$, $Y_i \in \mathbb{R}$ and $g \in C^{0,\lambda}(\bar{\Omega})^r$, where*

$$\int_\Omega (g, y)\, dx = \int_{\partial\Omega} (\vartheta, y)\, d\omega \quad \text{for all } y \in \mathcal{Y}(\Omega)^r \qquad (3.51)$$

and $Hg_{\mathcal{M}} = 0$ for the extension $g_{\mathcal{M}} \in C^{0,\lambda}(\mathcal{M})^r$ of g.

The boundary problem

$$\triangle f = g \text{ in } \Omega$$

$$\nu \wedge f = \xi \ \text{ and } \nu \wedge \delta f = \vartheta \text{ on } \partial\Omega$$

$$Y_i[f] = Y_i, \ i = 1, \dots, B_{n-r},$$

is called $\triangle_r$-Dirichlet problem, or explicitly $\triangle_{\mathcal{D},\Omega}(g, \xi, \vartheta, \{Y_i | i = 1, \dots, B_{n-r}(\Omega)\})$.

b) The $\boldsymbol{\Delta_r}$***-Neumann problem****: We assume that* $\xi^* \in C_\nu^{2,\lambda}(\partial\Omega)^{r-1}$, $\vartheta^* \in C_\nu^{1,\lambda}(\partial\Omega)^r$, $Z_i \in \mathbb{R}$ *and* $g^* \in C^{0,\lambda}(\bar{\Omega})^r$, *where*

$$\int_\Omega (g^*, z)\, dx = \int_{\partial\Omega} (\vartheta^*, z)\, d\omega \quad \textit{for all } z \in \mathcal{Z}(\Omega)^r \tag{3.52}$$

and $Hg^*_{\mathcal{M}} = 0$ *for the extension* $g^*_{\mathcal{M}} \in C^{0,\lambda}(\mathcal{M})^r$ *of* g^*.

The boundary problem

$$\triangle f^* = g^* \textit{ in } \Omega$$

$$\nu\lrcorner\, f^* = \xi^* \textit{ and } \nu\lrcorner\, d f^* = \vartheta^* \textit{ on } \partial\Omega$$

$$Z_i[f^*] = Z_i,\ i = 1, \ldots, B_r,$$

is called $\triangle_r$*-Neumann problem, or explicitly* $\triangle_{\mathcal{N},\Omega}(g^*, \xi^*, \vartheta^*, \{Z_i | i = 1, \ldots, B_r(\Omega)\})$.

Each $\triangle_r$-Neumann problem can be formulated as a $\triangle_{n-r}$-Dirichlet problem and vice versa by means of the Hodge mapping $*$, as shown in the next paragraph.

We obtain for $\triangle$ applied to $f^* := * f$:

$$g^* = \triangle f^* = \triangle * f = * \triangle f = * g, \tag{3.53}$$

and for the boundary values

$$\xi^* = \nu\lrcorner\, f^* = \nu\lrcorner * f = (-1)^r * (\nu \wedge f) = (-1)^r * \xi \tag{3.54}$$

and

$$\vartheta^* = \nu\lrcorner\, d f^* = \nu\lrcorner\, d * f = (-1)^{r-1} \nu\lrcorner\, (*\delta f) = *\vartheta. \tag{3.55}$$

For the integrability condition one can easily infer

$$\begin{aligned} \int_\Omega (g, y)\, dx &= \int_\Omega \left(*^{-1} g^*, *^{-1} z\right) dx = \int_\Omega (g^*, z)\, dx = \int_{\partial\Omega} (\vartheta^*, z)\, d\omega \\ &= \int_{\partial\Omega} \left(*^{-1}\vartheta^*, *^{-1} z\right) d\omega = \int_{\partial\Omega} (\vartheta, y)\, d\omega. \end{aligned} \tag{3.56}$$

The conditions $Y_i[f] = Y_i$ and $Z_i[f^*] = Z_i$ are called topological conditions. It is convenient to express our $\triangle_r$-problems by operator equations, where the boundary conditions are included, but the topological conditions are excluded. Then, our Dirichlet and Neumann problems define $\triangle_r$-Dirichlet and $\triangle_r$-Neumann operators, respectively. The dimension of the $\triangle_r$-Dirichlet operator's kernel is equal to B_{n-r}, and the dimension of the $\triangle_r$-Neumann one is correspondingly equal to B_r.

We will show that the $\triangle_r$-Dirichlet and $\triangle_r$-Neumann problems can be solved, and that these solutions are uniquely.

Theorem 3.8 *Let* $0 < r < n$, $0 < \lambda < 1$, $g \in C^{0,\lambda}(\bar{\Omega})^r$, $\xi \in C^{2,\lambda}_{\tau}(\partial\Omega)^{r+1}$, $\vartheta \in C^{1,\lambda}_{\tau}(\partial\Omega)^r$, $g^* \in C^{0,\lambda}(\bar{\Omega})^r$, $\xi^* \in C^{2,\lambda}_{\nu}(\partial\Omega)^{r-1}$, *and* $\vartheta^* \in C^{1,\lambda}_{\nu}(\partial\Omega)^r$.

Then the according solutions of the Dirichlet problem

$$\triangle_{\mathcal{D},\Omega}(g, \xi, \vartheta, \{Y_i | i = 1, \dots, B_{n-r}(\Omega)\})$$

and the Neumann problem

$$\triangle_{\mathcal{N},\Omega}(g^*, \xi^*, \vartheta^*, \{Z_i | i = 1, \dots, B_r(\Omega)\})$$

belong to $C^{2,\lambda}(\bar{\Omega})^r$. *Furthermore, these solutions are uniquely determined.*

Proof: The proof with regard to $\Omega \subset\subset \mathbb{R}^n$, $n > 2$, can be taken from [2]. Here we will generalize it and adapt it to the assumptions of this theorem.

Green's formula implies that solutions can differ merely by Dirichlet fields. But the only Dirichlet r-field with vanishing values Y_i is the null r-form.

As $Hg_{\mathcal{M}} = 0$ for the extension $g_{\mathcal{M}} \in C^{0,\lambda}(\mathcal{M})^r$ of g is presupposed, a differential form $f_{\mathcal{M}} := Gg_{\mathcal{M}} \in C^{2,\lambda}(\mathcal{M})^r$ exist, which fulfills

$$\triangle f_{\mathcal{M}} = g_{\mathcal{M}}. \tag{3.57}$$

Instead of the original problem

$$\triangle_{\mathcal{D},\Omega}(g, \xi, \vartheta, \{Y_i | i = 1, \dots, B_{n-r}(\Omega)\}),$$

we may confine our task to solve

$$\triangle_{\mathcal{D},\Omega}(0, \tilde{\xi}, \tilde{\vartheta}, \{\tilde{Y}_i | i = 1, \dots, B_{n-r}(\Omega)\}),$$

where

$$\tilde{\xi} = \xi - \nu \wedge f_{\mathcal{M}}, \;\; \tilde{\vartheta} = \vartheta - \nu \wedge \delta f_{\mathcal{M}}, \;\text{ and } \tilde{Y}_i = Y_i - Y_i[f_{\mathcal{M}}]. \tag{3.58}$$

The condition (3.51) provides

$$\int_{\partial\Omega} \left(\tilde{\vartheta}, y\right) d\omega = 0 \quad \text{for all } y \in \mathcal{Y}(\Omega)^r. \tag{3.59}$$

Let $\tilde{\xi} \in C^{2,\lambda}_{\tau}(\partial\Omega)^{r+1}$ and $\tilde{\vartheta} \in C^{1,\lambda}_{\tau}(\partial\Omega)^r$ be fixed. If ϕ satisfies

$$\left(\frac{1}{2} I + W_{2,1,\lambda,r}\right)\phi = \tilde{\theta} := \begin{pmatrix} \nu \lrcorner \tilde{\xi} \\ \nu \lrcorner \tilde{\vartheta} \end{pmatrix}, \tag{3.60}$$

then $f_{\phi|\Omega}$ is a solution of $\triangle_{\mathcal{D},\Omega}(0, \tilde{\xi}, \tilde{\vartheta}, \{Y_i | i = 1, \dots, B_{n-r}(\Omega)\})$, where $Y_i = Y_i[f_{\phi|\Omega}]$.

We set $\varrho^* \in \Gamma(\mathcal{M}, \Lambda^r T^*(\mathcal{M}))$, $\zeta^* \in \Gamma(\mathcal{M}, \Lambda^{r-1} T^*(\mathcal{M}))$, and

$$\rho^* := \begin{pmatrix} \varrho^* \\ \zeta^* \end{pmatrix}.$$

Fredholm theory shows that

$$\tilde{\theta} \in \mathcal{R}\left(\frac{1}{2} I + W_{2,r}\right) \quad \text{if} \quad \int_{\partial\Omega} (\tilde{\theta}, \rho^*)\, d\omega = 0 \text{ for all } \rho^* \in \mathcal{N}\left(\frac{1}{2} I + W^*_{2,r}\right). \tag{3.61}$$

Since $\rho^* \in \mathcal{N}\Big(\frac{1}{2} I + W^*_{2,r}\Big)$, the assigned form $f^*_{\theta^*}|_{\hat{\Omega}}$ belongs to

$$\triangle_{\mathcal{N},\hat{\Omega}}(0, 0, 0, \{Z_i | i = 1, \dots, B_r(\hat{\Omega})\}).$$

Hence, $f^*_{\rho^*}$ is a Neumann field in $\hat{\Omega}$.

From equation (3.30) and

$$\left(\frac{1}{2} I + U^*_{2,r}\right) \varrho^* = 0, \tag{3.62}$$

we conclude that $d f^*_{\rho^*}|_\Omega$ is a Dirichlet field.

As $\triangle f^*_{\rho^*} = 0$, the condition

$$\int_{\partial\Omega} \left(\nu \lrcorner\, d f^*_{\rho^*}, \hat{z}\right) d\omega = 0 \quad \text{for all } \hat{z} \in \mathcal{Z}(\hat{\Omega})^r \tag{3.63}$$

follows. Additionally, [8], *Satz 7.2* and [3], *Theorem 2*, yield

$$-\nu \lrcorner\, d f^*_{\rho^*} = \rho^*. \tag{3.64}$$

This, together with (3.63) implies that $\varrho^* = 0$ and thus $\zeta^* \in \mathcal{N}\Big(\frac{1}{2} I + U^*_{2,r-1}\Big)$.

Since the equation

$$\int_{\partial\Omega} (\nu \lrcorner\, \tilde{\vartheta}, \zeta^*)\, d\omega = 0 \quad \text{for all } \zeta^* \in \mathcal{N}\left(\frac{1}{2} I + U^*_{2,r-1}\right) \tag{3.65}$$

corresponds to

$$\int_{\partial\Omega} (\tilde{\vartheta}, y)\, d\omega = 0 \quad \text{for all } y \in \mathcal{Y}(\Omega)^r, \tag{3.66}$$

and the integrability condition (3.51) can be taken into account, the Dirichlet problem

$$\triangle_{\mathcal{D},\Omega}(g, \xi, \vartheta, \{Y_i | i = 1, \dots, B_{n-r}(\Omega)\})$$

has a unique solution in $C^{2,\lambda}(\bar{\Omega})^r$.

The Neumann and the Dirichlet problem are connected to each other by the Hodge mapping. □

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